2024

MATH1180 2024/25

Coursework 1

COMPUTATIONAL METHODS AND NUMERICAL TECHNIQUES

JAKE CUNNINGHAM - 001211278

Student Banner ID

0	0	1	2	1	1	2	7	8

Ignore first 3 digits and replace any 0's with 5

2	1	1	2	7	8

Order from Smallest to Largest

1	1	2	2	7	8

P= 1 Q=1 R=2 S=2 T=7 U=8

a = 1 + (2x2)

a = 1 + 4

a = 5

b = 1 + (0.5 x 7)

b = 1 + 3.5

b = 4.5

c = 2 + (0.5 x 8)

c = 2 + 4

c = 6

d = 1 + (2 x 2)

d = 1 + 4

d = 5

e = 2 + 7

e = 9

f = 2(1 + 8)

f = 2 x 9

f = 18

Q1

(i)

BE – Is possible as the number of columns on B matches the number of rows on E, the matrix will be 3x2

EB – Is not possible as the number of rows on E is greater than the number of columns on B

V'E – Is possible, as v will transpose from 2x1 to 1x2, this means that it will have the same number of rows as E has columns. This will make the result a 1x3 matrix.

E^{-1} – Is not possible as the matrix is not a square, where all rows and columns are equal.

(ii)

Transformed E is the sum of C multiplied by E

E' = C x E

$$E' = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} * \begin{pmatrix} 5 & 4.5 & 6 \\ 5 & 9 & 18 \end{pmatrix}$$

$$E' = \begin{pmatrix} 25 & 22.5 & 30 \\ 25 & 45 & 90 \end{pmatrix}$$

The transformation matrix C scales the object E by a factor of 5, on both the X and Y axis. Meaning the resulting object will be 5 times greater than the original while retaining the shape and proportions.

(iii)

$$A^{-1} = \frac{1}{\det{(A)}} * \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det(A) = (a * d) - (b * d)$$

$\det(A) = (5 * 5) - (4.5 * 6)$ = 25 – 27 = -2

$$\det(A) = -2$$

$$A^{-1} = \frac{1}{-2} * \begin{pmatrix} 5 & -4.5 \\ -6 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{5}{-2} & \frac{-4.5}{-2} \\ \frac{-6}{-2} & \frac{5}{-2} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2.5 & 2.25 \\ 3 & -2.5 \end{pmatrix}$$

(iv)

$Ax = v$

$x = A^{-1}v$

$x = \begin{pmatrix} -2.5 & 2.25 \\ 3 & -2.5 \end{pmatrix} * \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

$x = \begin{pmatrix} -13 \\ 16 \end{pmatrix}$

(v)

$B - \lambda I$

$B - \lambda I = \begin{pmatrix} -\lambda & -1 \\ 9 & 18 - \lambda \end{pmatrix}$

$det \begin{pmatrix} -\lambda & -1 \\ 9 & 18 - \lambda \end{pmatrix} = (-\lambda)(18 - \lambda) - (-1)(9)$

$(18 - \lambda) = \ -18\lambda + \lambda^2$

$(-1)(9) = -9$

$det = \lambda 2 - 18\lambda + 9$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda = \dfrac{-(-18) \pm \sqrt{(-18)^2 - 4(1 * 9)}}{(2 * 1)}$

$= \dfrac{18 \pm \sqrt{324 - 36}}{2}$

$= \dfrac{18 \pm \sqrt{288}}{2}$

$= \dfrac{18 \pm 16.971}{2}$

$\dfrac{18 + 16.971}{2} = 17.485$

$\dfrac{18 - 16.971}{2} = 0.515$

$\lambda 1 \approx 17.485 \ and \ \lambda 2 \approx 0.515$

(vi)

When matrix B is applied to its eigenvectors, the eigenvectors are stretched or shrunk, but their direction remains the same.

(vii)

The values will remain the same as it is a scalar matrix, this means that because both values are the same on the diagonals and when scaled the vectors increase uniformly.

(viii)

(a)

$$cos(\theta) = \frac{u \cdot w}{|u||w|}$$

First find $u \cdot w$

$$u \cdot w = \begin{pmatrix} 5 \\ 4.5 \\ 6 \end{pmatrix}\begin{pmatrix} 5 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 25 \\ 40.5 \\ 108 \end{pmatrix} = 173.5$$

Then find magnitude of u

$$|u| = \sqrt{5^2} + 4.5^2 + 6^2 = \sqrt{25} + 20.25 + 36 = \sqrt{81.25}$$

$$|u| = 9.014$$

Followed by the magnitude of w

$$|w| = \sqrt{5^2} + 9^2 + 18^2 = \sqrt{25} + 81 + 324 = \sqrt{430}$$

$$|w| = 20.736$$

$$|u||w| = 9.014 * 20.736 = 186.914$$

Put all that together

$$cos(\theta) = \frac{173.5}{186.914}$$

$$cos(\theta) = 0.928$$

To calculate in radian

$$\theta = \cos^{-1}(0.928) = 0.382$$

The angle is 0.382 Radians

(b)

The two vectors are orthogonal if $u * z = 0$

$$u * z = \begin{pmatrix} 5 \\ 4.5 \\ 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = \begin{pmatrix} 5 \\ 4.5 \\ 6a \end{pmatrix}$$

$$5 + 4.5 + 6a = 9.5 + 6a$$

Solve for a

$$9.5 + 6a = 0$$

$$6a = 9.5$$

$$a = \frac{9.5}{6}$$

$$a = 1.583$$

Q2

(i)

Homogeneous coordinates in computer graphics and motion pictures provide a unified framework for transformations like translation, rotation, scaling, and projection. It can avoid division operations which improves efficiency and stability especially for complex shapes. Homogeneous coordinates also facilitate perspective projections, this provides lifelike depth in scenes. (Skala, 2008)

(ii)

A (5, 4.5)

B (6, 5)

C (9, 18)

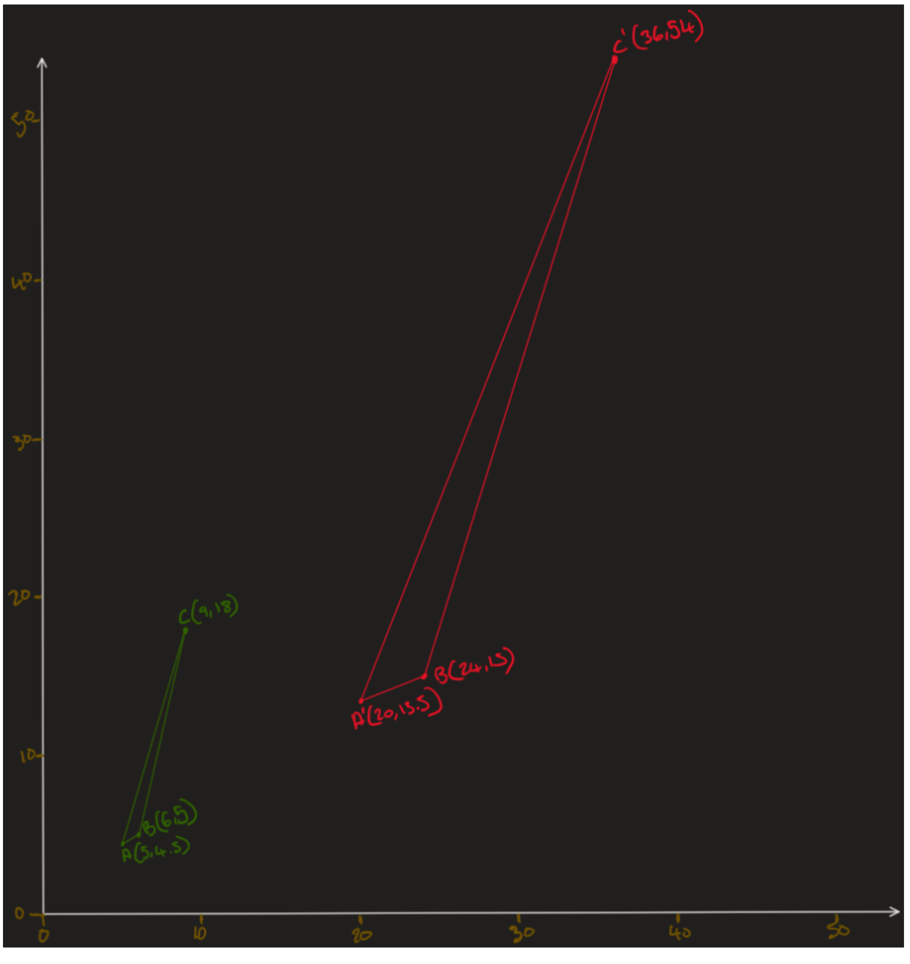
$New\ x = Old\ x * \ scale\ factor\ in\ x$

$New\ y = Old\ y * \ scale\ factor\ in\ y$

$A' = (5 * 4, 4.5 * 3) = (20, 13.5)$

$B' = (6 * 4, 5 * 3) = (24, 15)$

$C' = (9 * 4, 18 * 3) = (36, 54)$



(iii)

$rotated\ x = x' * \cos(\theta) - y' * \sin(\theta)$

$rotated\ y = x' * \sin(\theta) + y' * \cos(\theta)$

$\theta = 45°$

$\cos(45°) = \sin(45°) = \frac{\sqrt{2}}{2} = 0.7071$

Rotate A’ (20, 13.5)

$rotated\ x = (20 * 0.7071) - (13.5 * 0.7071) = 4.596$

$rotated\ y = (20 * 0.7071) + (13.5 * 0.7071) = 23.687$

$A''(4.596, 23.687)$

Rotate B’ (24, 15)

$rotated\ x = (24 * 0.7071) - (15 * 0.7071) = 6.364$

$rotated\ y = (24 * 0.7071) + (15 * 0.7071) = 27.577$

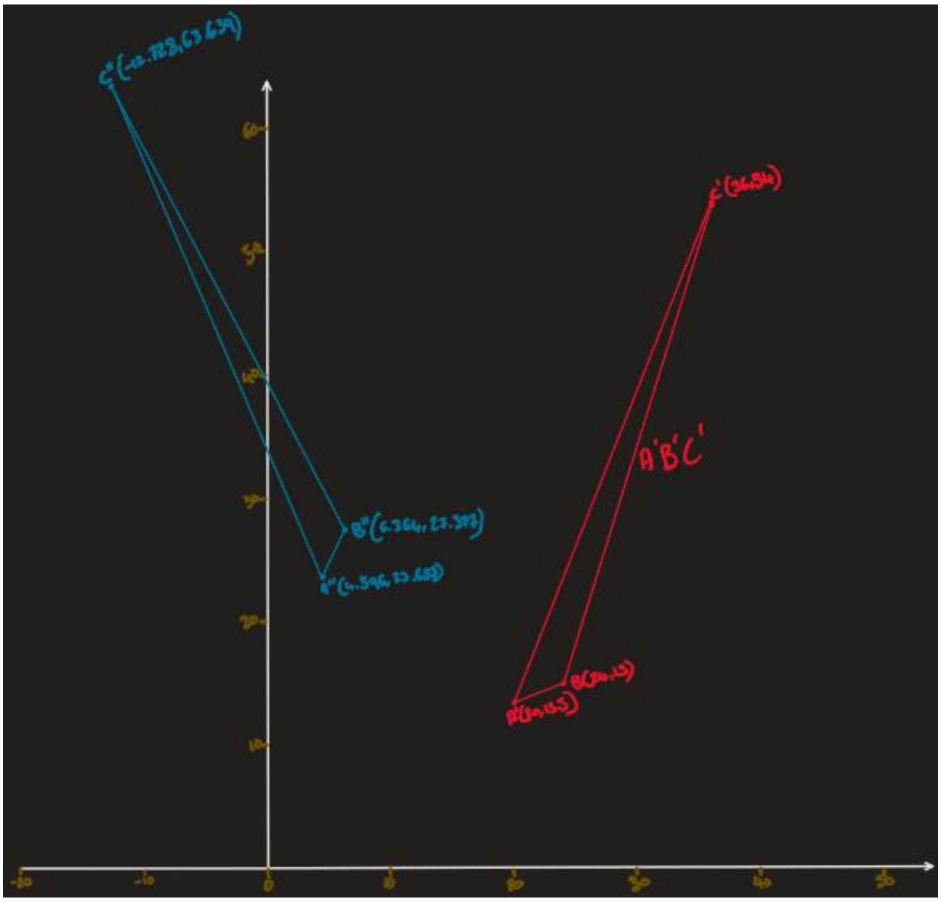
$B''(6.364, 27.577)$

Rotate C’ (36, 54)

$rotated\ x = (36 * 0.7071) - (54 * 0.7071) = -12.728$

$rotated\ y = (36 * 0.7071) + (54 * 0.7071) = 63.639$

$C''(-12.728, 63.639)$



(iv)

P_1 = (5, 4.5, 6)

P_2= (5, 9, 18)

P_3= (-5, -4.5, -6)

Projection matrix for $yz = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

P'$_1$ $= Pprojection * \begin{pmatrix} 5 \\ 4.5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 5 \\ 4.5 \\ 6 \end{pmatrix}$

$(0 * 5) + (0 * 4.5) + (0 * 6) = 0$

$(0 * 5) + (1 * 4.5) + (0 * 6) = 4.5$

$(0 * 5) + (0 * 4.5) + (1 * 6) = 6$

P'$_1$ $= \begin{pmatrix} 0 \\ 4.5 \\ 6 \end{pmatrix}$

P'$_2$ $= Pprojection * \begin{pmatrix} 5 \\ 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 5 \\ 9 \\ 18 \end{pmatrix}$

$(0 * 5) + (0 * 9) + (0 * 18) = 0$

$(0 * 5) + (1 * 9) + (0 * 18) = 9$

$(0 * 5) + (0 * 9) + (1 * 18) = 18$

P'$_2$ $= \begin{pmatrix} 0 \\ 9 \\ 18 \end{pmatrix}$

P'$_3$ $= Pprojection * \begin{pmatrix} -5 \\ -4.5 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} -5 \\ -4.5 \\ -6 \end{pmatrix}$

$(0 * -5) + (0 * -4.5) + (0 * -6) = 0$

$(0 * -5) + (1 * -4.5) + (0 * -6) = -4.5$

$(0 * -5) + (0 * -4.5) + (1 * -6) = -6$

P'$_3$ $= \begin{pmatrix} -0 \\ -4.5 \\ -6 \end{pmatrix}$

P_1 = (0, 4.5, 6)

P_2= (0, 9, 18)

P_3= (0, -4.5, -6)

(v)

Perspective projection can be used in computer graphics, engineering, and photography; it depicts a 3D object onto a 2D plane while attempting to preserve depth. Unlike other projections where parallel lines remain parallel, perspective projection makes these converge at a vanishing point, attempting to be as similar as possible to what seen by the human eye. For example, a long, continuous road. As the road stretches into the distance, the sides of the road seem to converge, even though they are parallel. (Gambetta, 2021)

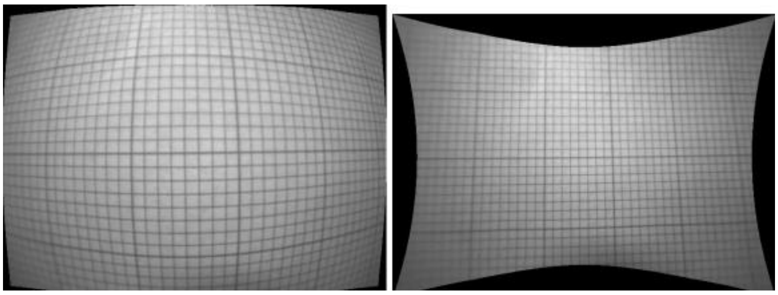


Fig. 1 Shows the difference between Perspective projection (Right) and Stereographic projection (Left) (Fleck, 1995)

Whereas stereographic projection retains the parallelism of lines. It distorts the object shape, which can be observed in Fig. 1. The underlying difference represented from the diagram is related to the fact that while perspective projection allows lines to remain straight converging at a point, on stereographic projection, the lines seemingly start to curve outwards from one another.

Assuming the camera is at point (0, 0, 0). For a 3D point, (x, y, z) the coordinates for a projection place are calculated as:

$$x_{projected} = \frac{d * x}{z} \qquad y_{projected} = \frac{d * y}{z}$$

Here, d denotes the distance between the camera and the projection plane, while z represents the depth of a certain point in 3D. This eventually leads to the 2D projection coordinates of a given point, (xprojected, yprojected). When the position of that point moves farther, the projected coordinates become shrunken to simulate exactly how, in real life, things would look smaller if their distances are further. (Gambetta, 2021)

Q3

(i)

$$\frac{dy}{dx} = \frac{d}{dx}(ax^4) + \frac{d}{dx}(bx^2) - \frac{d}{dx}(8cx) + \frac{d}{dx}(d) + \frac{d}{dx}(e\, sin\, x)$$

$$\frac{d}{dx}(ax^4) = 4ax^3$$

$$\frac{d}{dx}(bx^2) = 2bx$$

$$\frac{d}{dx}(-8cx) = -8c$$

$\frac{d}{dx}(d) = 0$ (Constant)

$$\frac{d}{dx}(e \sin x) = e \cos x$$

$$\frac{dy}{dx} = 4a^3 + 2bc - 8c + e \cos x$$

$$\frac{dy}{dx} = 4(5)x^3 + 2(4.5)x - 8(6) + 9 \cos x$$

$$\frac{dy}{dx} = 20x^3 + 9x - 48 + 9 \cos x$$

$$x = \frac{\pi}{4}$$

$$x^3 = \left(\frac{\pi}{4}\right)^3 = \frac{\pi^3}{4}$$

$$20x^3 = 20 * \frac{\pi^3}{4} = \frac{20\pi^3}{64} = \frac{5\pi^3}{16}$$

$$9x = 9 * \frac{\pi}{4} = \frac{9\pi}{4}$$

-48 remains the same

$$\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$9 * cos\frac{\pi}{4} = \frac{9\sqrt{2}}{2}$$

$$\frac{dy}{dx} = \frac{5\pi^3}{16} + \frac{9\pi}{4} - 48 + \frac{9\sqrt{2}}{2}$$

(ii)

Forward Difference Approximation Formula

$$\frac{dy}{dx} \approx \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

Backward Difference Approximation Formula

$$\frac{dy}{dx} \approx \frac{y(x) - y(x - \Delta x)}{\Delta x}$$

Central Difference Approximation Formula

$$\frac{dy}{dx} \approx \frac{y(x + \Delta x) - y(x - \Delta x)}{2\Delta x}$$

$x = \frac{\pi}{4}$ $\Delta x = \frac{\pi}{10}$

$$x + \Delta x = \frac{\pi}{4} + \frac{\pi}{10} = \frac{5\pi}{20} + \frac{2\pi}{20} = \frac{7\pi}{20}$$

$$x - \Delta x = \frac{\pi}{4} - \frac{\pi}{10} = \frac{5\pi}{20} - \frac{2\pi}{20} = \frac{3\pi}{20}$$

Forward Difference

$$\frac{dy}{dx} \approx \frac{\sin\left(\frac{7\pi}{20}\right) - sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{10}}$$

$$sin\left(\frac{7\pi}{20}\right) \approx 0.8910$$

$$sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 0.7071$$

Therefore, the Forward Difference is: $\frac{dy}{dx} \approx \frac{0.8910 - 0.7071}{\frac{\pi}{10}} \approx \frac{0.1839}{0.3141} \approx 0.5854$

Backward Difference

$$\frac{dy}{dx} \approx \frac{\sin\left(\frac{\pi}{4}\right) - sin\left(\frac{3\pi}{20}\right)}{\frac{\pi}{10}}$$

$$sin\left(\frac{3\pi}{20}\right) \approx 0.4540$$

$$sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 0.7071$$

Therefore, the Backward Difference is: $\frac{dy}{dx} \approx \frac{0.7071 - 0.4540}{\frac{\pi}{10}} \approx \frac{0.2531}{0.3141} \approx 0.8057$

Central Difference

$$\frac{dy}{dx} \approx \frac{\sin\left(\frac{7\pi}{20}\right) - sin\left(\frac{3\pi}{20}\right)}{2 * \frac{\pi}{10}}$$

$$sin\left(\frac{7\pi}{20}\right) \approx 0.8910$$

$$sin\left(\frac{3\pi}{20}\right) \approx 0.4540$$

Therefore, the Central Difference is: $\frac{dy}{dx} \approx \frac{0.8910-0.4540}{2\frac{\pi}{10}} \approx \frac{0.4560}{0.6283} \approx 0.7258$

Forward Difference: ≈ 0.5854

Backward Difference: ≈ 0.8057

Central Difference: ≈ 0.7258

(iii)

$$e = O(\Delta x^2) = -\frac{\Delta x^2}{12} f'''(x)$$

$$\Delta x = \frac{\pi}{10} \approx 0.3142$$

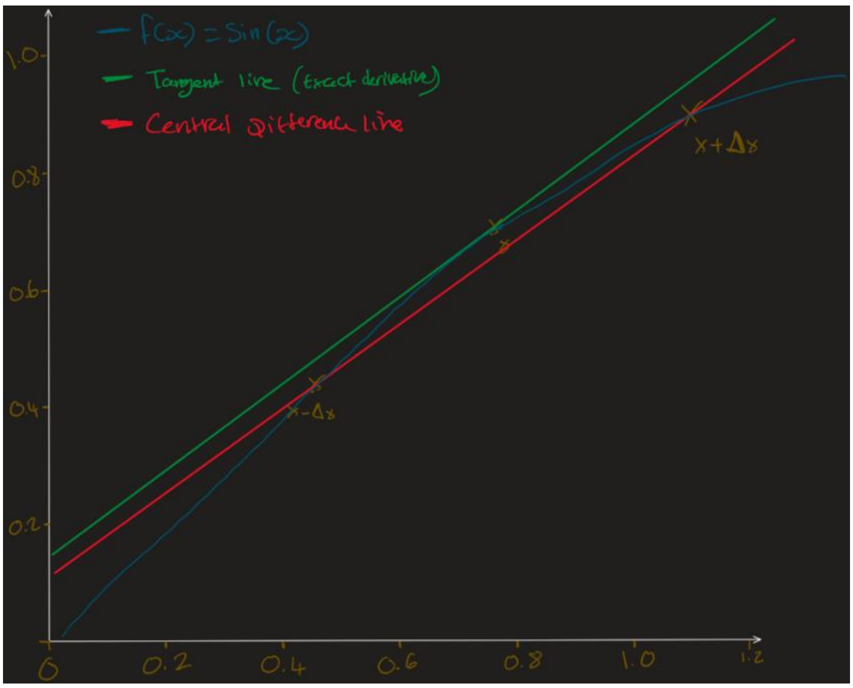
$$\Delta x^2 = 0.0987$$

$$f'''(x) = -\cos(x) = -0.7071$$

$$e = -\frac{0.0987}{12} * (-0.7071) \approx 0.00582$$

The truncation error is approximately 0.00582

(iv)



References

Fleck, M. M., 1995. Perspective projection: the wrong imaging model. *Department of Computer Science, University of Iowa,* pp. 1-27.

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